Here $X(x)$ is an arbitrary function; k, a and b are constants.

The solution of Eq. (1) under conditions (7) is obtained in [1] by the Liapunov-Charpy method [2].

By virtue of the Jacobi theorem [3], the resulting total integral (1) can be used to find the solution of the associated canonical system of differential equations. This system yields the differential Eq.

$$\frac{d^2r}{dt^2} = -2\frac{\partial F}{\partial r}$$

Thus, the latter equation can be solved for a function F satisfying condition (6).

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CYCLES AND QUASI-INDICES OF SINGULAR POINTS OF CONSERVATIVE SYSTEMS

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Henri Poincaré [1] noted that closed trajectories (cycles) investigated, in the whole, play a role roughly analogous to that of singular points in the study of the behavior of trajectories in the small.

However, the problem of finding the cycles is in itself quite difficult. Among the criteria of existence of periodic trajectories for two-dimensional systems we must first of all note the criteria based on a consideration of vector field rotation (the indices of the Poincaré singular points).

A sufficient criterion for the existence of periodic trajectories on a plane based on the so-called ring principle whereby the velocity vector on the boundary of the domain is everywhere directed into or out of the ring was pointed out by Bendixon [2 and 3].

There exist still other methods of investigation in the whole, among them the method of Liapunov functions [4].

The criterion of existence of periodic trajectories for conservative systems in the so-called invertible case, which is based on a consideration of the variation of the action integral, was set forth by Whittaker [5]. Our study of cycles for conservative systems is based on a different principle, and specifically on the study of quasi-indices as structural characteristics of singular points [6].

1. Let a mechnical system with two degrees of freedom move in a conservative field with the potential $V(q_1, q_2)$, and let its Hamiltonian $H(p_i, q_i)$ be of the form

$$H(p_i, q_i) = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2)$$

In the invertible case of the Hamilton equation which we are considering

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (i = 1, 2) \tag{1.1}$$

admit of the energy integral $H(p_i, q_i) = h$. We shall assume that the potential $V(q_i)$ has isolated singular points O_j ($j = 1, \ldots, k$) at which the function $V(q_i)$ becomes infinite.

In many important cases, e. g. for fields formed by attracting (repelling) centers, the singular points O_j are simply poles of differing multiplicities.

Let us investigate the necessary conditions for the existence (nonexistence) of closed trajectories (cycles) for mechanical system (1.1) under consideration. To do this we write out the differential equation of the trajectories of system (1.1), making use of the principle of steady action in Jacobi form.

In the Cartesian system ($q_1 = x, q_2 = y$) the equation of trajectories is [7]

$$d\psi = \frac{\partial \Phi}{\partial y}dx - \frac{\partial \Phi}{\partial x}dy \qquad (\Phi = \ln \sqrt{2(h - V(x, y))}) \tag{1.2}$$

Here $\Psi(x, y) = \operatorname{arc\,tg} y'$ is the angle formed by the velocity vector v and the positive x-axis; h is the constant energy; the mass of the representing point M of the system is assumed to be equal to unity ($m = 1$).

Thus, our initial problem of finding the conditions of existence of the cycles of system (1.1) reduces to finding the conditions of existence of the corresponding solutions of the differential equation of the trajectories (1.2).

2. An important role in the qualitative theory of conservative system trajectories is played by the Pfaffian form

$$\omega = \frac{\partial \Phi}{\partial y}dx - \frac{\partial \Phi}{\partial x}dy \tag{2.1}$$

where $\Phi(x, y)$ is derived in accordance with (1.2) and where the variables dx and dy can be chosen arbitrarily and need not satisfy Hamilton system (1.1). Let us rewrite the form ω (2.1) differently. Since in moving along a certain contour (C) we have $dx = \cos(\mathbf{n}y)\,ds$, $dy = -\cos(\mathbf{n}x)\,ds$, where $\mathbf{n}$ is the direction of the interior normal to the contour (C) and ds is an arc element, it follows that

$$\omega = (\partial \Phi / \partial n)\, ds \tag{2.2}$$

In the case where the contour (C) is a whole cycle, so that dx and dy satisfy Hamilton system (1.1), by virtue of (1.2) and (2.1) we have $\omega = d\psi$, so that

$$\frac{1}{2\pi}\oint_{(C)} \frac{\partial \Phi}{\partial n} ds = 1 \tag{2.3}$$

Let the point O_j which we take as the origin be an isolated singular point of the potential $V(x, y)$ and therefore of the Hamiltonian H.

Let us consider the curvilinear integral of the differential form ω (2.1) taken along some closed (non-self-intersecting) contour (γ_j) surrounding the singular point O_j.

The limiting value of this integral divided by 2π, when the contour (γ_j) is contracted to zero without intersecting the singular point O_j, will be called the quasi-index J_j of the singular point O_j

$$J_j = \frac{1}{2\pi} \lim_{r \to 0} \oint_{(\gamma_j)} \omega \qquad (r = \sqrt{x^2 + y^2}) \tag{2.4}$$

The quasi-index for a regular point is always zero. This follows directly from (2, 2) and (2. 4) if we note that $\partial\Phi/\partial n$ is a bounded function in the neighborhood of a regular point. Depending on the structure of the singular points O_j, the quasi-indices J_j can assume arbitrary real values. This distinguishes them from Poincaré indices, which can assume integer values only.

Theorem 1. The quasi-index J_j depends neither on the choice of the curve (γ_j) nor on the method of taking the limit, provided that deformation of the contour (γ_j) does not make it intersect the singular point O_j.

In order to prove this theorem, let us construct the two arbitrary closed (non-self-intersecting) contours (C) and (γ_j) around the point O_j. Let the contour (γ_j) lie inside (C). The curvilinear integral of the differential form ω (2. 1) taken over the complex contour (Γ) $(C) + (\gamma_j)$ in such a way that the domain (σ^*) bounded by this contour remains on the left can be transformed into an integral over the area of (σ^*) with the aid of Green's theorem.

Then, taking the limit of contracting the contour (γ_j) to a point, by virtue of (2. 4) we obtain

$$J_j = \frac{1}{2\pi} \oint_{(C)} \omega + \frac{1}{2\pi} \iint_{(\sigma)} \Delta\Phi \, dx \, dy \tag{2.5}$$

Here (σ) is the domain bounded by the contour (C) and Δ is a Laplacian

Since the selection of a different (γ_j) around the singular point O_j and the taking of the limit as $r \to 0$ does not alter the right-hand side of (2. 5), this proves that the quasi-index J_j does not depend either on the choice of the contour (γ_j) or on the method of taking the limit. In fact, it depends only on the structure of the singular point O_j itself. The theorem has been proved.

Henceforth as our curves (γ_j) surrounding the singular points O_j we shall choose circles of small radius r with their centers at the points O_j.

We note that the double integral in the right-hand side of (2. 5) is singular, so that the quasi-index J_j has a finite value only if this integral exists.

In the presence of cycles we can establish a simple relationship between the Poincaré index I_j and the quasi-index J_j. In fact, let (C) be a cycle. Then, by virtue of (1. 2) and (2. 5) we obtain

$$J_j = I_j + \frac{1}{2\pi} \iint_{(\sigma)} \Delta\Phi \, dx \, dy \tag{2.6}$$

The Poincaré index for a cycle is equal to unity $(I_j = 1)$. This leads to the following fundamental relation:

$$J_j = 1 + \frac{1}{2\pi} \iint_{(\sigma)} \Delta\Phi \, dx \, dy \tag{2.7}$$

Relation (2. 7) can be readily generalized for the case where the cycle (C) surrounds k singular points O_j $(j = 1, 2, \ldots, k)$.

Surrounding the singular points O_j with the circles (γ_j) of small radius r and analyzing the contour $(\Gamma) = (C) + (\gamma_1) + \ldots + (\gamma_k)$ as above for the case of one singular point, we obtain

$$J = 1 + \frac{1}{2\pi} \iint_{(\sigma)} \Delta\Phi \, dx \, dy \qquad \left(J = \sum_{j=1}^{k} J_j\right) \tag{2.8}$$

Here J is the sum of quasi-index J_j of the singular points O_j lying inside the cycle

(C) ; (σ) is the domain bounded by the contour (C) minus the singular points O_j .

In the case where the differential form ω (2.1) is a total differential [8] so that the condition $\Delta\Phi(x, y) = 0$ is fulfilled and where cycles are present, by virtue of (2.6) the quasi-index J_j coincides with the value of Poincaré index I_j (i e $J_j = I_j = 1$).

We note that by virtue of (2.1) and (2.4) the quasi-index J_j can be written in complex form

$$J_j = -\operatorname{Re}\left(\lim_{|z|\to 0} \frac{1}{2\pi i} \oint_{(\gamma_j)} \Omega\, dz\right) \qquad \left(\Omega = \frac{\partial\Phi}{\partial x} - i\frac{\partial\Phi}{\partial y}\right) \tag{2.9}$$

Let $\Omega(z)$ be an analytic function of the complex variable $z = x + iy$, so that $\Delta\Phi(x, y) = 0$ by virtue of the Cauchy-Riemann conditions.

If the singular point $z = z_j$ is a pole of multiplicity n , then the Loran series expansion is of the form

$$\Omega(z) = \frac{a_{-n}}{z^n} + \cdots + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$$

so that, by virtue of (2.9), the quasi-index $J_j = -\operatorname{Re}(a_{-1})$.

3. The quasi-indices of singular points are quite simple for the important class of fields with potentials of the form $V = V(r)$.

Let the point $O(r = 0)$ which we take as our origin, be a singular point of the potential $V(r)$.

Then, by virtue of (2.2) and (2.4), the quasi-index J of the point O is

$$J = -\lim\left(r\,\frac{\partial\Phi}{\partial r}\right) \qquad (r \to 0) \tag{3.1}$$

Hence, the quasi-index J has a finite value different from zero only if the expansion of $\Phi(r)$ in the neighborhood of zero is of the form

$$\Phi(r) = a_0 \ln r + a_1 r + a_2 r^2 + \ldots \qquad (a_0 \neq 0)$$

so that, by virtue of (3.1), the quasi-index $J = -a_0$. For example, for the potential

$$V(r) = A_1 r^{-1} + A_2 r^{-2} + \cdots + A_n r^{-n}$$

by virtue of (1.2) we obtain

$$\Phi(r) = \frac{1}{2}\ln\left(\frac{2}{r^n}\left(hr^n - \sum A_k r^{n-k}\right)\right) = -\frac{n}{2}\ln r + F(r)$$

where $F(r)$ is an entire function of r . In this case $a_0 = -n/2$ and the quasi-index $J = n/2$. We can point out a case where the quasi-index J does not have a finite value. This is true, in fact, when the point $O(r = 0)$ is essentially singular and when the expansion of $V(r)$ in the neighborhood of zero is of the form

$$V(r) = A_1 r^{-1} + A_2 r^{-2} + \ldots$$

This is the form, for example, of the potential of a ring [9] and of the potential of a spheroid for points lying in the equatorial plane [10]. In these cases we have

$$J = \lim\left({}^1/_2 n\right) = \infty \qquad (n \to \infty)$$

For a logarithmic potential $V = A \ln r$, the quasi-index J of the singular point $r = 0$ is equal to zero. Hence, in the neighborhood of zero we have the expansion

$$\Phi(r) = {}^1/_2 \ln\left(2\left(h - A\ln r\right)\right) = {}^1/_2 \ln\ln{}^1/r + F(r)$$

where $F(r)$ is an entire function Hence,

$$J = -{}^1/_2 \lim\left(r \ln\ln{}^1/r\right) = 0 \qquad (r \to 0)$$

4. Considering motion on a projective plane, we introduce the notion of the quasi-index J^* of an infinitely distant point. Let the point $O(r=0)$ which we take as our origin be a singular point. Let us surround the point O with some closed (non-self-intersecting) contour (γ). The limiting value of the integral

$$J^* = \frac{1}{2\pi} \lim_{r\to\infty} \oint_{(\gamma)} \omega \qquad \left(\omega = -\frac{\partial \Phi}{\partial n}\right) \tag{4.1}$$

will be called the quasi-index of an infinitely distant point. Here n is the direction of the exterior normal to the contour (γ). We move along the contour (γ) in such a way that the domain (σ_1) containing the infinitely distant point remains on the left (i.e. clockwise around the point O).

Specifically, for a central field with the potential $V=V(r)$, on surrounding the point $O(r=0)$ with a circle (γ) of radius r, we obtain

$$J^* = \lim\left(r\frac{\partial \Phi}{\partial r}\right) \qquad (r\to\infty) \tag{4.2}$$

Thus, for example, for a field with the potential $V=-A/r^n$ $(A,\ n>0)$ for a constant energy $h=0$ the quasi-index J^* of an infinitely distant point is

$$J^* = \frac{1}{2}\lim\left(r\frac{\partial}{\partial r}\ln\left(2\left(h+\frac{A}{r^n}\right)\right)\right) = -\frac{n}{2}$$

We recall that the quasi-index of the singular point $O(r=0)$ in this case was $J=n/2$.

We note that the same results (and in particular Formula (4.2)) can be obtained by setting $r=1/\rho$ and converting the potential $V=-A/r^n$ into $V_1=-A\rho^n$. In this case the infinitely distant point and the singular point $O(r=0)$ change places.

5. The function $\delta(x, y) = (1/2\pi)\triangle\Phi(x, y)$ will be called the density of the distribution $\Phi(x, y)$. By virtue of (1.2) this function can be written as

$$\delta(x, y) = -\frac{(h-V)\triangle V + (V_x^2+V_y^2)}{4\pi(h-V)^2} \tag{5.1}$$

while for central fields, where $V=V(r)$,

$$\delta(r) = \frac{1}{2\pi r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) \qquad (\Phi = \ln\sqrt{2(h-V(r))}) \tag{5.2}$$

The line at which the density $\delta(x, y)$ changes sign will be called the zero density line. The equation of this line is

$$F(x, y, h) = (h-V)\triangle V + (V_x^2+V_y^2) = 0 \tag{5.3}$$

It can turn out that a zero density line does not exist for the given potential $V(x, y)$ and for certain values of the parameter h. In this case the density δ will be a function which does not change its sign over the domain (σ).

The density $\delta(x, y)$ will also be of constant sign wherever the condition $\Delta V \geq 0$ is fulfilled, and it will be sign $(\delta) = -1$ for any value of the constant energy h.

It can also happen that the line of zero density is closed (it can even be self-intersecting) or it breaks down into n closed branches $(C_1), \ldots, (C_n)$. In the latter case the density $\delta(x, y)$ will not change its sign inside each of the curves (C_i). This follows from the continuity of the density $\delta(x, y)$ on passage through the zero density line.

For example, let us consider the zero density line in the problem of two stationary centers, assuming that the attracting masses are equal and that the law of attraction is arbitrary.

Assuming that the attracting centers lie at the points $O_1(x_1 = {}^1/_2 a,\ y_1 = 0)$ and

O_2 ($x_2 = -\frac{1}{2}a$, $y_2 = 0$), we write out the potential V of the field in question

$$V = -\left(\frac{A}{r_1^n} + \frac{A}{r_2^n}\right) \qquad (A,\ n > 0,\ r_{1,2} = \sqrt{(x \pm a/2)^2 + y^2})$$

where, by virtue of (5.1), the density $\delta(x, y)$ is given by

$$\delta(x, y) = \frac{Ahn^2(r_1^{n+2} + r_2^{n+2}) + A^2a^2n^2}{4\pi(h - V)^2(r_1r_2)^{n+2}}$$

Hence, zero density lines do not exist for hyperbolic-parabolic types of motion ($h \geq 0$). In the case of elliptic motion ($h < 0$) the zero density line does exist, and its equation is of the form

$$r_1^{n+2} + r_2^{n+2} = B \qquad (B = -Aa^2/h) \tag{5.4}$$

In the case of Newtonian attraction ($n = 1$) we obtain $r_1^3 + r_2^3 = B$.

Simple analysis shows that this implies fulfillment of the inequality

$$r_1r_2 < C^2 \quad (C = (B/2)^{1/3})$$

so that the line of zero density lies inside the Cassinian oval $r_1r_2 = C^2$.

6. The conditions of existence (absence) of cycles can be formulated in terms of the weights $P(\sigma)$ of the function $\Phi(x, y)$,

$$P = \iint_{(\sigma)} \delta(x, y)\, dx\, dy \qquad \left(\delta = \frac{1}{2\pi}\Delta\Phi\right) \tag{6.1}$$

where (σ) is the domain bounded by the contour (C) and $\delta(x, y)$ is, as before, the density of the distribution $\Phi(x, y)$. Basic relation (2.8) here becomes

$$1 - J = -P \qquad (J = J_1 + J_2 + \ldots + J_k) \tag{6.2}$$

The following theorems are valid.

Theorem 2. Let the sum of quasi-indices J_j of the singular points O_j in the domain (σ) under consideration assumes one of the following values:

$$\text{a)}\ -\infty < J < 1, \qquad \text{b)}\ J = 1, \qquad \text{c)}\ 1 < J < \infty \tag{6.3}$$

Then, the sufficient condition for the absence of cycles within (σ) is, that the line of zero density (or any part of it) does not exist in (σ) and that the weights P of the function $\Phi(x, y)$ obey the corresponding relations

$$\text{a)}\ P \geqslant 0, \qquad \text{b)}\ P \neq 0, \qquad \text{c)}\ P \leqslant 0 \tag{6.4}$$

This theorem can also be formulated in terms of the densities $\delta(x, y)$ (see [6]).

Theorem 3. Let one of the conditions (6.3) be fulfilled in the domain (σ) under consideration. Then, if no line of zero density (or any part of it) exists in this domain, then the condition that the weight P assumes the corresponding sign given by

$$\text{a)}\ P < 0, \qquad \text{b)}\ P = 0, \qquad \text{c)}\ P > 0 \tag{6.5}$$

is necessary but not sufficient for the existence of cycles.

Theorem 4. Let the density $\delta(x, y)$ be equal to zero everywhere in the domain (σ) except for the singular points. Then, if no combination of values of the quasi-indices J_j of the singular points O_j lying in (σ) is equal to unity, this is a sufficient condition for the absence of cycles in the domain (σ).

We note that the equality to zero of the density $\delta = (\frac{1}{2}\pi)\Delta\Phi$ in some domain corresponds to the condition of the total differential of the Pfaffian form ω (2.1).

Theorem 5. Let the density $\delta(x, y)$ changes its sign in the closed domain (σ) which is a circle of radius R with its center at the isolated singular point $O(r = 0)$ (i.e.

let a zero density line exist).

Then, if the weight P is equal to zero for any circle of radius $r \leq R$ with its center at the point O which lies entirely in the domain (σ), and if the quasi-index J_0 of the singular point $O(r = 0)$ is not equal to unity, then this is a sufficient condition for the absence of cycles in the domain (σ).

The above Theorems can be proved by considering basic relation (6.2).

7. As we know [11], periodic motions for conservative systems do not exist in isolation, i.e. if there exists a periodic motion for some value of the constant energy h, then periodic motion also exists for a constant h_1 close to h.

Let us investigate the existence of several cycles in the neighborhood of an isolated singular point for the same value of the constant energy h.

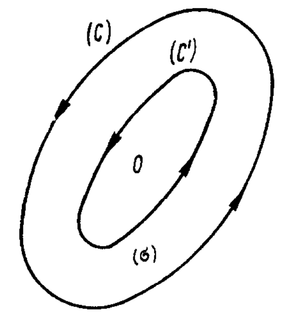

Fig. 1

Let two cycles (C) and (C') exist and form a ring (Fig. 1) for a given potential $V(x, y)$ and for some value of h.

The curvilinear integral of the Pfaffian form (2.1) taken over the complex contour $(\Gamma) = (C) + (C')$ can be transformed by means of Green's theorem into an area integral

$$\oint_{(C)} \omega - \oint_{(C')} \omega = -\iint_{(\sigma)} \triangle\Phi \, dx \, dy$$

Since (C) and (C') are cycles by hypothesis, by virtue of (2.2) and (2.3) we obtain

$$P = \frac{1}{2\pi} \iint_{(\sigma)} \triangle\Phi \, d\sigma = 0 \qquad (7.1)$$

i.e. the equality to zero of the weight (P) of the function Φ over the area of the ring (σ) is a necessary condition for the existence of the cycles (C) and (C').

Thus, we have the following theorem.

Theorem 6. The sufficient condition for the absence of two cycles, one within the other at the given value of h is, that the density $\delta(x, y)$ does not change its sign.

8. Let us consider the conditions of existence of circular cycles in a central force field with the potential $V = V(r)$.

Writing out the differential equation of the trajectories in polar coordinates [12],

$$-r\left[r^2 + \left(\frac{\partial r}{\partial \alpha}\right)^2\right]\frac{\partial V}{\partial r} + 2(h - V)\left[r^2 + 2\left(\frac{\partial r}{\partial \alpha}\right)^2 - r\frac{\partial^2 r}{\partial \alpha^2}\right] = 0 \qquad (8.1)$$

and setting the derivatives of r with respect to α equal to zero, after dividing through by a nonzero factor r^2 we obtain the following condition on the contour of the circular cycle (C):

$$Y(r) = 2h \qquad (Y(r) = r\partial V / \partial r + 2V) \qquad (8.2)$$

Condition (8.2) can be derived directly by means of the d'Alambert principle. This is easy to see when the derivation is carried out using the energy integral.

The values r_j for the cycles (C_j) corresponding to the given value of the constant energy h are the roots of the equation $Y(r) = 2h$. It can turn out that several cycles are associated with the same value of the constant h. As an example let us consider the superposition of the potentials of two attracting centers

$$V(r) = \frac{A}{r^{n-1}} + \frac{B}{r^n} \qquad (A, B < 0, \ n \geqslant 1) \qquad (8.3)$$

In this case we have

$$Y(r) = \frac{A(3-n) + B(2-n)}{r^n} \tag{8.4}$$

Analysis of the curves $Y = Y(r)$ (Fig. 2) for various n shows that for a given value of the constant energy h two cycles can exist only for $2 < n < 3$. For a given n from the interval $2 < n < 3$ the values $h_1 = 0$ and $h_2 = \frac{1}{2}Y(r_2)$ are bifurcational.

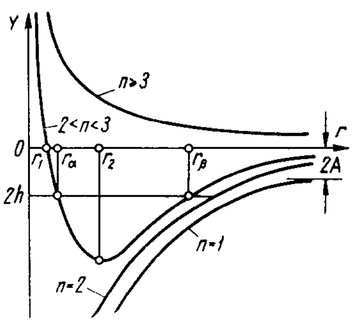

Fig. 2

Specifically, for $0 \leq h < \infty$ there exists one cycle of radius $r(0 < r \leq r_1)$, while for $h_2 < h < h_1$ there exist two cycles (C_α) and (C_β) with the radii r_α $(r_1 < r_\alpha < r_2)$ and r_β $(r_2 < r_\beta < \infty)$. For $h = h_2$ there exists one cycle with the radius $r = r_2$.

Here
$$r_1 = \frac{B(n-2)}{A(3-n)}, \qquad r_2 = \frac{n}{n-1} r_1,$$
$$Y(r_2) = \frac{B(n-2)}{(n-1)r_2^n} \tag{8.5}$$

We note that the necessary condition for the existence of two cycles (7.1) is completely fulfilled in in this case.

9. Let us consider the behavior of the trajectories of a conservative system in the neighborhood of a quiescent point, and, specifically, on the generation of cycles by the equilibrium state.

The coordinates of the quiescent point $O(x_0, y_0)$ can be found from the steady-state condition for the potential
$$\frac{\partial V(x, y)}{\partial x} = 0, \qquad \frac{\partial V(x, y)}{\partial y} = 0 \tag{9.1}$$

Without limiting generality we can assume that $V(x_0, y_0) = 0$ for the equilibrium position $x_0 = y_0 = 0$.

The expansion of $V(x, y)$ in the neighborhood of zero is of the form
$$V = V_n(x, y) + V_{n+1}(x, y) + \cdots \qquad (n \geqslant 2) \tag{9.2}$$
where $V_k(x, y)$ is a homogeneous form in the variables x and y of degree k.

In order to find the quasi-index J of the quiescent point we make use of (2.4), (2.1), and also of the value of $\Phi(x, y)$ in accordance with (1.2). This yields
$$J = \frac{1}{2\pi} \lim_{r \to 0} \oint_{(\gamma)} \frac{-V_y\, dx + V_x\, dy}{2(h - V)} \tag{9.3}$$
where (γ) is a circle of small radius r with its center at the quiescent point, while the constant energy h must be taken equal to zero. This follows from the energy interval, since by hypothesis $V_0 = V(0, 0) = 0$.

On the contour (γ) we have $x = r\cos\alpha$, $y = r\sin\alpha$, $dx = -y d\alpha$, $dy = x d\alpha$ so that by virtue of Euler's theorem on homogeneous functions we obtain
$$-V_y\, dx + V_x\, dy = (yV_y + xV_x)\, d\alpha = (nV_n + (n+1)V_{n+1} + \cdots)\, d\alpha$$

In polar coordinates $V_k(r, \alpha) = r^k \sigma_k(\alpha)$, where $\sigma_k(\alpha)$ is a homogeneous trigonometric form of degree k. Hence, by virtue of the familiar theorem on taking a limit in the integrand, we obtain
$$J = \frac{1}{2\pi} \lim_{r \to 0} \int_0^{2\pi} \frac{(n\sigma_n(\alpha) + r(n+1)\sigma_{n+1}(\alpha) + \cdots)\, d\alpha}{-2(\sigma_n(\alpha) + r\sigma_{n+1}(\alpha) + \cdots)} = -\frac{n}{2} \tag{9.4}$$
where n is the degree of the lowest-degree form V_n in expansion (9.2) of $V(x, y)$.

Limiting ourselves from now on to the lowest-degree form $V_2(n = 2)$ in expansion (9.2) of V (V_2 determines the sign of $V(x, y)$ for small values of x and y), we obtain the following expansion in the neighborhood of zero:

$$V(x, y) = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2) \tag{9.5}$$

$$A = \left(\frac{\partial^2 V}{\partial x^2}\right)_0, \qquad B = \left(\frac{\partial^2 V}{\partial x\, \partial y}\right)_0, \qquad C = \left(\frac{\partial^2 V}{\partial y^2}\right)_0 \tag{9.6}$$

From Expression (5.1) for the density δ we find that

$$\operatorname{sign}\delta = -\operatorname{sign}\mu, \qquad \mu = (h - V)\triangle V + (V_x^2 + V_y^2)$$

Evaluating the above derivatives and setting $h = 0$, we obtain

$$\mu = {}^1/_2\,(ay^2 + 2bxy + cx^2) \tag{9.7}$$

$$a = C^2 - AC + 2B^2, \qquad b = B(A + C), \qquad c = A^2 - AC + 2B^2 \tag{9.8}$$

The discriminant of the resulting form μ (9.7) is

$$\Delta = b^2 - ac = (AC - B^2)((A - C)^2 + 4B^2) \tag{9.9}$$

Let us prove the following theorems.

Theorem 7. Let the potential energy V (9.5) in the equilibrium position have an isolated minimum, i.e. let the condition of the Lagrange theorem be fulfilled so that the system is in Liapunov-stable equilibrium.

The necessary conditions for the existence of cycles are then fulfilled in the neighborhood of this point, and cycles can, in fact, exist.

Proof. Since the discriminant of quadratic form (9.5) is negative ($B^2 - AC < 0$, A, $C > 0$), while $\triangle V = (A + C) > 0$ and $h - V > 0$ by virtue of the energy integral, the density $\delta(x, y)$ (5.1) will be negative (sign $\delta = -1$), so that the weight $P < 0$.

Converting to the polar coordinates (r, α) and setting $h > 0$ (for $h \leq 0$ motion is not possible in the domain $V(x, y) > 0$), we can compute the value of the quasi-index J (9.3) of the quiescent point. We have

$$J = \frac{1}{2\pi}\lim_{r\to 0} r^2 \int_0^{2\pi} \frac{(A\cos^2\alpha + 2B\sin\alpha\cos\alpha + C\sin^2\alpha)\, d\alpha}{2\,(h - {}^1/_2\, r^2\,(A\cos^2\alpha + 2B\sin\alpha\cos\alpha + C\sin^2\alpha))} = 0$$

Since $J = 0$, $P < 0$ in the stable equilibrium position, it follows by Theorem 3 that the necessary conditions for the existence of cycles are fulfilled.

Theorem 8. Let the potential energy $V(x, y)$ (9.5) in the equilibrium position have an isolated maximum and let the equilibrium position be unstable by Liapunov's inversion theorem [13]. Then cycles cannot exist in the neighborhood of the equilibrium position for the same constant energy $h = 0$.

Proof. Let $V_0 = V(0, 0) = 0$, and therefore $h = 0$, in the unstable equilibrium position. Considering the motion for the constant energy $h = 0$, we can find the quasi-index J_0 (9.4) of the quiescent point $O(r = 0)$.

Setting $n = 2$, we obtain $J_0 = -1$.

Since the discriminant of quadratic form (9.5) $B^2 - AC < 0$ (A, $C < 0$), it follows that the discriminant $\Delta = b^2 - ac$ of the quadratic form μ (9.7) is positive by virtue of (9.9). Hence, the form μ will change its sign and this, in turn, implies that the density $\delta(x, y)$ will behave in the same manner, since sign $\delta = -$ sign μ.

Converting to the polar coordinates (r, α) and setting the constant energy $h = 0$, we can show that the weight P of the function Φ is equal to zero over the area of the circle

(σ) of arbitrary radius r.

To do this we need merely prove that the integral

$$K = \int_0^{2\pi} \frac{(\dot{a}\sin^2\alpha + 2b\sin\alpha\cos\alpha + c\cos^2\alpha)\,d\alpha}{(A\cos^2\alpha + 2B\sin\alpha\cos\alpha + C\sin^2\alpha)}$$

vanishes. Introducing the complex variable $\tau = e^{i\alpha}$, we obtain

$$K = M \oint_{(\gamma)} \frac{\psi(\tau)\,d\tau}{(\tau-\tau_1)^2(\tau-\tau_2)^2} \qquad \left(M = \text{const}, \quad \tau_{1,2} = \frac{-(A+C) \pm 2\sqrt{AC-B^2}}{A-C-2iB}\right)$$

where integration is carried out over the unit circle (γ), and

$$\psi(\tau) = (c - a - 2ib)\tau^2 + 2(a + c)\tau + (c - a + 2ib)$$

Omitting the intervening computations, we note that $|\tau_1\tau_2| = 1$, $|\tau_2| < 1$, $\text{Res}(\tau_2) = 0$, so that by virtue of the Cauchy residue theorem we find that $K = 2\pi i M \,\text{Res}(\tau_2) = 0$. This proves the statement that $P = 0$.

Thus, the conditions of Theorem 5 are fulfilled in this case ($P = 0$ for any $r > 0$, and $J_0 = -1$), so that the sufficient conditions for the absence of cycles are fulfilled. The theorem has been proved.

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